NODAL NONCOMMUTATIVE JORDAN ALGEBRAS AND SIMPLE LIE ALGEBRAS OF CHARACTERISTIC p

BY

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A noncommutative Jordan algebra A over a field F is a nodal algebra [9](2) in case every element of A may be written in the form $\alpha 1+z$ where α is in F, 1 is the unity element of A, and z is nilpotent, while the set N of nilpotent elements is not a subalgebra of A. F is necessarily of characteristic $p>0$.

L. A. Kokoris gave in [5] the first examples of simple nodal noncommutative Jordan algebras. Generalizing these, he has constructed in [7] a class K of nodal noncommutative Jordan algebras A of dimension p^n, and has proved [6; 7] that each simple nodal noncommutative Jordan algebra of characteristic $p \neq 2$ is in K. Although not all of the algebras in K are simple, it turns out that those algebras with which we are principally concerned here are indeed simple.

Any algebra A of dimension p^n in K may be represented as follows: let $B_n=F[x_1, \cdots, x_n]$, $x_i^p=0$, be a truncated polynomial ring with partial differentiation operators $\partial/\partial x_i$. Write $f\cdot g$ for the commutative associative product in B_n. Then A is the same vector space as B_n, but multiplication in A is defined by

$$fg = f\cdot g + \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij}, \qquad c_{ij} = -c_{ji}, \tag{1}$$

where the c_{ij} $(=-c_{ji})$ are elements in B_n which are arbitrary except for the proviso that at least one of them has an inverse (equivalently, at least one of them is not in the radical N of B_n). Then necessarily $n \geqq 2$. Also we note that

$$c_{ij} = \frac{1}{2}\,[x_i, x_j], \qquad i, j = 1, 2, \cdots, n. \tag{2}$$

This representation of an algebra A in K is not unique. It is well-known [4, p. 108; or 8, §4.5, Proposition 6] that any representatives in B_n of the elements of any basis of the n-dimensional space $N/N\cdot N$ will serve for the

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x_i $(i=1, \cdots, n)$. By the chain rule $\partial f/\partial x_i = \sum_{k=1}^n (\partial f/\partial y_k)\cdot(\partial y_k/\partial x_i)$, one obtains

$$fg = f\cdot g + \sum_{k,l=1}^{n} \frac{\partial f}{\partial y_k}\cdot\frac{\partial g}{\partial y_l}\cdot d_{kl}$$

with

$$d_{kl} = \frac{1}{2}\,[y_k, y_l] = \sum_{i,j=1}^{n} \frac{\partial y_k}{\partial x_i}\cdot\frac{\partial y_l}{\partial x_j}\cdot c_{ij}, \tag{3}$$

where $y_1, \cdots, y_n$ are any elements of B_n whose residue classes form a basis for $N/N\cdot N$ over F.

In this paper we study the derivations of the algebras in K. We are led to relationships between some of these algebras and recently discovered simple Lie algebras of characteristic p. In particular, we obtain an intrinsic characterization of the simple Lie algebras V_r of A. A. Albert and M. S. Frank [1]. Also we display each of the simple Lie algebras $L(G, \delta, f)$ of characteristic $\neq 2$ defined by Richard Block [2] as an ideal in the derivation algebra of a suitably chosen simple nodal noncommutative Jordan algebra A. *We assume characteristic $p>2$ throughout.*

1. **Derivations.** Since $c_{ij} = -c_{ji}$ in (1), we have $(fg+gf)/2 = f\cdot g$. That is, the commutative algebra A^+ attached to A is B_n itself.

Clearly any derivation of A is also a derivation of A^+. But the derivations of B_n are well-known [4, p. 107]:

$$f \to \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\cdot a_k$$

for arbitrary elements a_k in B_n. Hence, if D is a derivation of A, we have

$$fD = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\cdot a_k \tag{4}$$

for all f in A and for certain elements a_k $(=x_kD)$ in A $(k=1, \cdots, n)$. Occasionally we shall employ the notation $D=(a_1, \cdots, a_n)$ used in [3] and [1].

Suppose that D is given by (4). Then D is a derivation of A^+, and also

$$\frac{\partial h}{\partial x_i}D - \frac{\partial(hD)}{\partial x_i} = \sum_k \left(\frac{\partial^2 h}{\partial x_k \partial x_i}\cdot a_k - \frac{\partial^2 h}{\partial x_i \partial x_k}\cdot a_k - \frac{\partial h}{\partial x_k}\cdot\frac{\partial a_k}{\partial x_i}\right)$$
$$= -\sum_k \frac{\partial h}{\partial x_k}\cdot\frac{\partial a_k}{\partial x_i}$$

for any h in A. Hence

$$(fg)D - (fD)g - f(gD) = (f\cdot g)D - (fD)\cdot g - f\cdot(gD)$$
$$+ \sum_{i,j}\left\{\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij}\right)D - \frac{\partial(fD)}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial(gD)}{\partial x_j}\cdot c_{ij}\right\}$$
$$= \sum_{i,j}\left\{\frac{\partial f}{\partial x_i}D\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij} + \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}D\cdot c_{ij} + \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij}D - \frac{\partial(fD)}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial(gD)}{\partial x_j}\cdot c_{ij}\right\}$$
$$= \sum_{i,j,k}\left(-\frac{\partial f}{\partial x_k}\cdot\frac{\partial a_k}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot c_{ij} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_k}\cdot\frac{\partial a_k}{\partial x_j}\cdot c_{ij} + \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\frac{\partial c_{ij}}{\partial x_k}\cdot a_k\right)$$
$$= \sum_{i,j}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\left\{\sum_k\left(\frac{\partial c_{ij}}{\partial x_k}\cdot a_k - \frac{\partial a_i}{\partial x_k}\cdot c_{kj} - \frac{\partial a_j}{\partial x_k}\cdot c_{ik}\right)\right\}.$$

Hence D is a derivation of A if and only if

$$\sum_{k=1}^{n}\left(\frac{\partial c_{ij}}{\partial x_k}\cdot a_k + \frac{\partial a_i}{\partial x_k}\cdot c_{jk} + \frac{\partial a_j}{\partial x_k}\cdot c_{ki}\right) = 0 \tag{5}$$

for $1 \leqq i, j \leqq n$. The equations (5) are redundant for $j \geqq i$, so we have proved

THEOREM 1. *Let A be a nodal noncommutative Jordan algebra in K, multiplication being defined by* (1). *Then a mapping D on A is a derivation of A if and only if D has the form* (4) *for elements $a_1, \cdots, a_n$ in A satisfying* (5) *for $1 \leqq i < j \leqq n$.*

Solution of the equations (5) in general seems a formidable task. Even a preliminary simplification of the problem by using (3) to normalize the c_{ij} seems exceedingly complex in the general situation. In the next two sections we treat two special cases: (i) n arbitrary, but all of the c_{ij} in $F1$; (ii) $n=2$, but c_{ij} arbitrary in A.

2. **Algebras defined by skew-symmetric bilinear forms.** Suppose that all of the c_{ij} are in $F1$. That is, $c_{ij} = \phi_{ij}1$, $\phi_{ij} = -\phi_{ji}$ in F, not all ϕ_{ij} zero, where $x_ix_j = x_i\cdot x_j + \phi_{ij}1$. There is a unique skew-symmetric bilinear form ϕ defined on the n-dimensional space $M = \sum_{i=1}^{n} Fx_i$ (equivalently, defined on $N/N\cdot N$) such that $\phi(x_i, x_j) = \phi_{ij}$. Then

$$fg = f\cdot g + \sum_{i,j}\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_j}\cdot\phi(x_i, x_j)1.$$

If x_i $(i = 1, \cdots, n)$ exist for which $c_{ij} = \phi(x_i, x_j)1$, we shall say that A is *defined by the skew-symmetric bilinear form* ϕ. (Since $N(N\cdot N) \subseteq N$ by [9, Theorem 5] we know that, if A is defined by different skew-symmetric bilinear forms, the forms are equivalent.) Let $2r$ $(2 \leqq 2r \leqq n)$ be the rank of ϕ.

A change of basis in M gives $\phi(x_i, x_{r+i}) = 1 = -\phi(x_{r+i}, x_i)$ for $i = 1, \cdots, r$; $\phi(x_i, x_j) = 0$ otherwise. The form ϕ is nondegenerate if and only if $n = 2r$.

We remark that an algebra A defined by a skew-symmetric bilinear form ϕ is simple if and only if ϕ is nondegenerate. For take the special basis above in M. If $n > 2r$, then $c_{i,2r+1} = \phi_{i,2r+1} 1 = 0$ for $i = 1, \cdots, n$, so that $fx_{2r+1} = f \cdot x_{2r+1} = x_{2r+1} f$ for every f in A by (1). It follows that $C = x_{2r+1} \cdot A$ is an ideal of A, $C \neq 0$, $C \neq A$. If $n = 2r$, we shall see in §4 that A is a simple nodal noncommutative Jordan algebra associated in a specific way with one of the algebras in a general class of simple Lie algebras of characteristic p. Of course the simplicity of A may also be established directly in this particular case.

If we take the special basis above for M in an algebra A defined by a skew-symmetric form, equations (5) become

$$\frac{\partial a_i}{\partial x_{r+j}} = \frac{\partial a_j}{\partial x_{r+i}}, \qquad i, j = 1, \cdots, r; \tag{6}$$

$$\frac{\partial a_{r+i}}{\partial x_j} = \frac{\partial a_{r+j}}{\partial x_i}, \qquad i, j = 1, \cdots, r; \tag{7}$$

$$\frac{\partial a_i}{\partial x_j} + \frac{\partial a_{r+j}}{\partial x_{r+i}} = 0, \qquad i, j = 1, \cdots, r; \tag{8}$$

and

$$\frac{\partial a_j}{\partial x_i} = 0, \qquad 1 \leqq i \leqq 2r;\ 2r + 1 \leqq j \leqq n. \tag{9}$$

Equations (6), (7), (8) involve only $a_1, \cdots, a_{2r}$. Hence the elements a_{2r+k} ($k = 1, \cdots, n - 2r$) are arbitrary in $F[x_{2r+1}, \cdots, x_n]$ by (9).

LEMMA 1. *Let f be in $B_n = F[x_1, \cdots, x_n]$. For any i ($1 \leqq i \leqq n$), write $R_i = F[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$ so that $B_n = R_i[x_i]$. Then there exists g in B_n satisfying*

$$\frac{\partial g}{\partial x_i} = f \tag{10}$$

if and only if f is of the form

$$f = \beta_0 + \beta_1 \cdot x_i + \beta_2 \cdot x_i^2 + \cdots + \beta_{p-2} \cdot x_i^{p-2}, \qquad \beta_j \in R_i. \tag{11}$$

If f satisfies (11), *there is always a solution g of* (10) *having the form*

$$g = x_i \cdot h, \qquad h \in B_n. \tag{12}$$

If g is one solution of (10), *then g^* is a solution if and only if*

$$g^* = g + h, \qquad h \in R_i. \tag{13}$$

The proof is straightforward.

Using the lemma, we give an inductive proof that the solutions of (6)–(9) are

$$(14)\qquad \begin{aligned} a_i &= \frac{\partial g}{\partial x_{r+i}} + \sigma_i \cdot x_{r+i}^{p-1} && \text{for } i = 1, \cdots, r, \\ a_{r+i} &= -\frac{\partial g}{\partial x_i} + \sigma_{r+i} \cdot x_i^{p-1} && \text{for } i = 1, \cdots, r, \end{aligned}$$

and

$$(15)\qquad a_{2r+k} = \sigma_{2r+k} \qquad \text{for } k = 1, \cdots, n-2r,$$

for arbitrary g in A and arbitrary σ_i, σ_{r+i}, σ_{2r+k} in $F[x_{2r+1}, \cdots, x_n]$. Clearly any elements $a_1, \cdots, a_n$ given by (14) and (15) satisfy equations (6)–(9). We are concerned only with establishing the fact that, if $a_1, \cdots, a_{2r}$ satisfy (6)–(8), then $a_1, \cdots, a_{2r}$ have the form given in (14).

Let $a_1 = \beta_0 + \beta_1 \cdot x_{r+1} + \cdots + \beta_{p-2} \cdot x_{r+1}^{p-2} + \beta_{p-1} \cdot x_{r+1}^{p-1}$, $\beta_j \in R_{r+1}$, so that, for any $k \neq r+1$,

$$\frac{\partial a_1}{\partial x_k} = \frac{\partial \beta_0}{\partial x_k} + \cdots + \frac{\partial \beta_{p-1}}{\partial x_k} \cdot x_{r+1}^{p-1}.$$

Then, putting $i=1$ in (6) and (8), we have

$$\frac{\partial \beta_{p-1}}{\partial x_k} = 0 \qquad \text{for } k = 1, \cdots, r, r+2, \cdots, 2r$$

by (11). Also $\partial \beta_{p-1}/\partial x_{r+1} = 0$. By the lemma there exist $g \in B_n$ and $\sigma_1 \in F[x_{2r+1}, \cdots, x_n]$ such that $a_1 = \partial g/\partial x_{r+1} + \sigma_1 \cdot x_{r+1}^{p-1}$. For any t $(2 \leqq t \leqq 2r)$ we assume that (14) holds for $i = 1, \cdots, t-1$. There are two cases to be considered: $t \leqq r$, and $t \geqq r+1$. If $t \leqq r$, we replace i in (6) by t, and obtain

$$\frac{\partial a_t}{\partial x_{r+j}} = \frac{\partial^2 g}{\partial x_{r+t} \partial x_{r+j}} \qquad \text{for } j = 1, \cdots, t-1,$$

and

$$\frac{\partial a_t}{\partial x_{r+j}} = \frac{\partial a_j}{\partial x_{r+t}} \qquad \text{for } j = t+1, \cdots, r.$$

It follows that $a_t = \partial g/\partial x_{r+t} + f$, where

$$f = \frac{\partial h}{\partial x_{r+t}} + \sigma_t \cdot x_{r+t}^{p-1}$$

for some h satisfying $\partial h/\partial x_{r+j} = 0$ for $j = 1, \cdots, t-1$, and some σ_t satisfying

$$\frac{\partial \sigma_t}{\partial x_k} = 0 \tag{16}$$

for $k = r+1, \cdots, 2r$. Write $g^* = g + h$. Then $\partial g^*/\partial x_{r+i} = \partial g/\partial x_{r+i}$ for $i=1, \cdots, t-1$, so that the inductive hypothesis remains valid with g^* replacing g. Also $a_t = \partial g^*/\partial x_{r+t} + \sigma_t \cdot x_{r+t}^{p-1}$ for σ_t satisfying (16) for $k=r+1, \cdots, 2r$. Putting $i=t$ in (8), we have

$$\frac{\partial \sigma_t}{\partial x_j} \cdot x_{r+t}^{p-1} = -\frac{\partial}{\partial x_{r+t}}\left(a_{r+j} + \frac{\partial g^*}{\partial x_j}\right),$$

so (16) holds also for $k=1, \cdots, r$. That is, $\sigma_t \in F[x_{2r+1}, \cdots, x_n]$, as desired. In the second case ($t \geqq r+1$), we write $t=r+s$. Then the assumption of the induction is that the first line of (14) holds for $i=1, \cdots, r$, and the second line for $i=1, \cdots, s-1$. Putting $j=s$ in (8), we have $a_{r+s} = -\partial g/\partial x_s + f$, where

$$\frac{\partial f}{\partial x_k} = 0 \tag{17}$$

for $k=r+1, \cdots, 2r$. Also $j=s$ in (7) yields (17) for $k=1, \cdots, s-1$, and

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_s}\left(a_{r+i} + \frac{\partial g}{\partial x_i}\right)$$

for $i=s+1, \cdots, r$. Hence $f = -\partial h/\partial x_s + \sigma_{r+s} \cdot x_s^{p-1}$ for some h satisfying $\partial h/\partial x_k = 0$ for $k=1, \cdots, s-1, r+1, \cdots, 2r$, and some σ_{r+s} satisfying

$$\frac{\partial \sigma_{r+s}}{\partial x_k} = 0 \qquad \text{for } k = 1, \cdots, 2r. \tag{18}$$

Write $g^* = g+h$. Then $\partial g^*/\partial x_k = \partial g/\partial x_k$ for $k=1, \cdots, s-1, r+1, \cdots, 2r$, so that the inductive hypothesis remains valid with g^* replacing g. Also $a_{r+s} = -\partial g^*/\partial x_s + \sigma_{r+s} \cdot x_s^{p-1}$ with $\sigma_{r+s} \in F[x_{2r+1}, \cdots, x_n]$ by (18). We have established equations (14).

Now any D given by (4), (14), and (15) determines g modulo $F[x_{2r+1}, \cdots, x_n]$ and the σ_j ($j=1, \cdots, n$) uniquely. Hence we have proved

THEOREM 2. *Let A be a nodal noncommutative Jordan algebra of dimension p^n which is defined by a skew-symmetric bilinear form ϕ of rank $2r$. Then the derivation algebra $D(A)$ of A has dimension*

$$p^n - p^{n-2r} + np^{n-2r} = p^{n-2r}(p^{2r} - 1 + n).$$

The x_i in A may be chosen so that the derivations D of A have the form (4) *with a_k as in* (14) *and* (15).

Let D be given by (4) and E by

$$fE = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot b_k \tag{19}$$

for b_k in A. Then it is well-known [4, p. 107] that $[D, E] = C = (c_1, \cdots, c_n)$ where

$$c_i = \sum_{j=1}^{n} \left(\frac{\partial a_i}{\partial x_j} \cdot b_j - \frac{\partial b_i}{\partial x_j} \cdot a_j \right), \qquad i = 1, \cdots, n. \tag{20}$$

It is easily seen that, in case $n > 2r$ above, there are many ideals in $D(A)$. If $n = 2r$, however, (that is, if A is defined by a nondegenerate ϕ) we have (14) with $\sigma_k \in F1$ $(=F)$ for $k = 1, \cdots, 2r$. Writing $D(g; \sigma_1, \cdots, \sigma_{2r})$ for D in (14), we obtain

$$[D(f; \rho_1, \cdots, \rho_{2r}), D(g; \sigma_1, \cdots, \sigma_{2r})] = D(h; 0, \cdots, 0) \tag{21}$$

from (20), where

$$\begin{aligned} h = \sum_{j=1}^{r} \Big\{ & (\rho_j \sigma_{r+j} - \rho_{r+j}\sigma_j) x_j^{p-1} \cdot x_{r+j}^{p-1} + \left(\frac{\partial f}{\partial x_j} \cdot \frac{\partial g}{\partial x_{r+j}} - \frac{\partial f}{\partial x_{r+j}} \cdot \frac{\partial g}{\partial x_j} \right) \\ & + \left(\sigma_{r+j} \frac{\partial f}{\partial x_{r+j}} - \rho_{r+j} \frac{\partial g}{\partial x_{r+j}} \right) \cdot x_j^{p-1} + \left(\sigma_j \frac{\partial f}{\partial x_j} - \rho_j \frac{\partial g}{\partial x_j} \right) \cdot x_{r+j}^{p-1} \Big\}. \end{aligned} \tag{22}$$

We recognize $D(g) = D(g; 0, \cdots, 0)$ as any element of the algebra V_{0r} of Albert and Frank [1, p. 127]. Let $\tilde{A}$ be the subspace of B_n consisting of all elements of B_n for which the coefficient of $x^{p-1} \cdot x_2^{p-1} \cdot \cdots \cdot x_n^{p-1}$ is zero. The $(p^{2r} - 2)$-dimensional simple Lie algebra V_r of Albert and Frank consists of all $D(g)$ with $g \in \tilde{A}$. It is known [1, pp. 127, 128] that V_r is the derived algebra V'_{0r} of V_{0r} and that $\partial f/\partial x_j \cdot \partial g/\partial x_{r+j} - \partial f/\partial x_{r+j} \cdot \partial g/\partial x_j$ is in $\tilde{A}$. It follows from (11) that $\partial f/\partial x_{r+j} \cdot x_j^{p-1} \in \tilde{A}$, etc. Hence only the first term within the braces in (22) could fail to be in $\tilde{A}$. If $r > 1$, h in (22) is in $\tilde{A}$, and $D(h) \in V_r$. Hence $V_r = V'_{0r} \subseteq D(A)' \subseteq V_r$ if $r > 1$. If $r = 1$, there exists h in (22) which is not in $\tilde{A}$, so that $D(A)' \subseteq V_{01}$ but $D(A)' \neq V_1$. However, $V_1 = V_1' \subseteq D(A)'' \subseteq V'_{01} = V_1$. Of course $D(A)'' = V_r$ for $r > 1$ also.

Thus we obtain the following intrinsic characterization of the $(p^{2r} - 2)$-dimensional simple Lie algebras V_r of Albert and Frank:

THEOREM 3. *A Lie algebra L is a simple algebra V_r if and only if there is a p^{2r}-dimensional (simple) nodal noncommutative Jordan algebra A defined by a nondegenerate skew-symmetric bilinear form such that $L \cong D(A)'$ in case $r > 1$, and $L \cong D(A)''$ in case $r = 1$ (actually $L \cong D(A)''$ in both cases).*

A second characterization of the algebras V_r results from the following observation concerning Lie-admissible algebras.

In any nonassociative algebra A, $xy = x \cdot y + [x, y]/2$. Therefore a linear transformation D on A is a derivation if and only if D is a derivation of both A^+ and A^-, where A^- is the anticommutative algebra attached to A having the product $[x, y]$. That is,

$$D(A) = D(A^+) \cap D(A^-). \tag{23}$$

Now in any nodal noncommutative Jordan algebra A in K

$$[f, g] = \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot 2c_{ij}, \tag{24}$$

and D in (4) is a derivation of A if and only if D is a derivation of A^-.

If A^- is a Lie algebra, then the mappings ad $g/2$ defined by

$$f \to \frac{1}{2}[f, g] = \sum_i \frac{\partial f}{\partial x_i} \cdot \left(\sum_j \frac{\partial g}{\partial x_j} \cdot c_{ij} \right) \tag{25}$$

are inner derivations of A^-, and therefore derivations of A since they are of the form (4) with

$$a_i = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} \cdot c_{ij}, \qquad i = 1, \cdots, n. \tag{26}$$

The set ad A of all inner derivations (25) of A^-, being an ideal of $D(A^-)$, is an ideal of $D(A)$ by (23).

We use the Jacobi identity

$$\begin{aligned}
0 &= [[f, g], h] + [[g, h], f] + [[h, f], g] \\
&= 2 \sum_{i,j} \left\{ \left[\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij}, h \right] + \left[\frac{\partial g}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} \cdot c_{ij}, f \right] + \left[\frac{\partial h}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} \cdot c_{ij}, g \right] \right\} \\
&= 4 \sum_{i,j,k,t} \left\{ \frac{\partial}{\partial x_t} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij} \right) \cdot \frac{\partial h}{\partial x_k} \cdot c_{tk} \right. \\
&\quad \left. + \frac{\partial}{\partial x_t} \left(\frac{\partial g}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} \cdot c_{ij} \right) \cdot \frac{\partial f}{\partial x_k} \cdot c_{tk} + \frac{\partial}{\partial x_t} \left(\frac{\partial h}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} \cdot c_{ij} \right) \cdot \frac{\partial g}{\partial x_k} \cdot c_{tk} \right\} \\
&= 4 \sum_{i,j,k,t} \left(\frac{\partial^2 f}{\partial x_t \partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot c_{ij} \cdot c_{tk} + \frac{\partial f}{\partial x_i} \cdot \frac{\partial^2 g}{\partial x_t \partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot c_{ij} \cdot c_{tk} \right. \\
&\quad + \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot \frac{\partial c_{ij}}{\partial x_t} \cdot c_{tk} + \frac{\partial^2 g}{\partial x_j \partial x_t} \cdot \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_i} \cdot c_{tk} \cdot c_{ji} + \frac{\partial g}{\partial x_k} \cdot \frac{\partial^2 h}{\partial x_i \partial x_t} \cdot \frac{\partial f}{\partial x_j} \cdot c_{kt} \cdot c_{ij} \\
&\quad + \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial c_{jk}}{\partial x_t} \cdot c_{ti} + \frac{\partial^2 h}{\partial x_t \partial x_i} \cdot \frac{\partial f}{\partial x_j} \cdot \frac{\partial g}{\partial x_k} \cdot c_{ij} \cdot c_{tk} + \frac{\partial h}{\partial x_k} \cdot \frac{\partial^2 f}{\partial x_i \partial x_t} \cdot \frac{\partial g}{\partial x_j} \cdot c_{kt} \cdot c_{ij} \\
&\quad \left. + \frac{\partial h}{\partial x_k} \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial c_{ki}}{\partial x_t} \cdot c_{tj} \right) \\
&= 4 \sum_{i,j,k} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot \frac{\partial h}{\partial x_k} \cdot \sum_t \left(\frac{\partial c_{ij}}{\partial x_t} \cdot c_{tk} + \frac{\partial c_{jk}}{\partial x_t} \cdot c_{ti} + \frac{\partial c_{ki}}{\partial x_t} \cdot c_{tj} \right)
\end{aligned}$$

to see that A^- is a Lie algebra if and only if

$$\sum_{t=1}^{n}\left(\frac{\partial c_{ij}}{\partial x_t}\cdot c_{tk} + \frac{\partial c_{jk}}{\partial x_t}\cdot c_{ti} + \frac{\partial c_{ki}}{\partial x_t}\cdot c_{tj}\right) = 0 \tag{27}$$

for $i, j, k=1, \cdots, n$. The equations (27) are redundant for $i \geqq j$ and for $j \geqq k$. Hence A^- is a Lie algebra if and only if (27) holds for $1 \leqq i < j < k \leqq n$. (It follows that A^- is a Lie algebra in case $n=2$.)

The equations (27) are obviously satisfied in any algebra A defined by a skew-symmetric bilinear form ϕ. Using the basis employed before in $M = \sum_i Fx_i$, we have ad $g/2$ in the form (4) with a_k given by (14) and (15) with $\sigma_1 = \cdots = \sigma_n = 0$. Therefore ad A is an ideal of dimension $p^{n-2r}(p^{2r}-1)$ in $D(A)$ which is of dimension $p^{n-2r}(p^{2r}-1+n)$. If ϕ is nondegenerate, so that $n=2r$, then ad A is the (nonsimple) algebra V_{0r} of all $D(g)$ for $g \in A$.

THEOREM 4. *A Lie algebra L is a simple Lie algebra V_r if and only if there is a p^{2r}-dimensional* (*simple*) *nodal noncommutative Jordan algebra A defined by a nondegenerate skew-symmetric bilinear form such that $L \cong$* (ad A)$'$.

3. **The case** $n=2$. Let A in K have least possible dimension p^2. Then it is known [5, §3] that A is simple. The vector space of A coincides with $B_2 = F[x_1, x_2]$, $x_1^p = x_2^p = 0$, and multiplication in A is defined by

$$fg = f\cdot g + \left(\frac{\partial f}{\partial x_1}\cdot\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\cdot\frac{\partial g}{\partial x_1}\right)\cdot c \tag{28}$$

where c has an inverse c^{-1} in B_2. Also (5) reduces to the single equation

$$\frac{\partial c}{\partial x_1}\cdot a_1 - \frac{\partial a_1}{\partial x_1}\cdot c + \frac{\partial c}{\partial x_2}\cdot a_2 - \frac{\partial a_2}{\partial x_2}\cdot c = 0,$$

which is equivalent to

$$\frac{\partial}{\partial x_1}(c^{-1}\cdot a_1) + \frac{\partial}{\partial x_2}(c^{-1}\cdot a_2) = 0. \tag{29}$$

If, given D in (4), we write $b\cdot D$ for the derivation

$$f \to \sum_{k=1}^{n}\frac{\partial f}{\partial x_k}\cdot(b\cdot a_k)$$

of B_n (that is, $b\cdot D = (b\cdot a_1, \cdots, b\cdot a_n)$), and if we write $\delta(D)$ for the *divergence* [3, p. 715]

$$\delta(D) = \sum_{k=1}^{n}\frac{\partial a_k}{\partial x_k},$$

then condition (29) becomes

$$\delta(c^{-1}\cdot D) = 0. \tag{30}$$

Hence D is a derivation of A if and only if $c^{-1}\cdot D$ is in the (p^2+1)-dimensional Lie algebra M_2 of derivations of B_2 having divergence zero [3]. But $D \leftrightarrow c^{-1}\cdot D$ is a vector space isomorphism, so the derivation algebra $D(A)$ is a (p^2+1)-dimensional algebra $D(A)=c\cdot M_2$. If $c=1$, so that A is defined by a (non-degenerate) form ϕ, then $D(A)=M_2$ consists of the derivations (4) of B_2 given by (14) with $r=1$, while $D(A)'=V_{01}$, and $D(A)''=V_1$ is simple. More generally two distinct situations arise, depending upon whether c^{-1} is in $\tilde{A}$ or not.

We digress momentarily to point out that (30) defines Lie algebras which generalize the algebras M_n of Frank [3], not only for $n=2$, but for general n. Let c be any invertible element of B_n, and let D range over the derivations of B_n satisfying (30). The set $c\cdot M_n$ of these derivations of B_n is a Lie algebra of dimension $(n-1)p^n+1$ since g in B_n implies

$$\delta(g\cdot[D, E]) = \delta(\delta(g\cdot D)\cdot E) - \delta(\delta(g\cdot E)\cdot D), \tag{31}$$

generalizing [3, Lemma 2]. For (20) gives

$$\begin{aligned}
&\delta(\delta(g\cdot D)\cdot E) - \delta(\delta(g\cdot E)\cdot D)\\
&= \sum_j \frac{\partial}{\partial x_j}\left\{\sum_i\left(\frac{\partial(g\cdot a_i)}{\partial x_i}\cdot b_j - \frac{\partial(g\cdot b_i)}{\partial x_i}\cdot a_j\right)\right\}\\
&= \sum_{i,j}\left(\frac{\partial g}{\partial x_i}\cdot\frac{\partial a_i}{\partial x_j}\cdot b_j + g\cdot\frac{\partial^2 a_i}{\partial x_j\partial x_i}\cdot b_j - \frac{\partial g}{\partial x_i}\cdot\frac{\partial b_i}{\partial x_j}\cdot a_j - g\cdot\frac{\partial^2 b_i}{\partial x_j\partial x_i}\cdot a_j\right)\\
&= \sum_i \frac{\partial g}{\partial x_i}\cdot c_i + g\cdot\sum_i\frac{\partial c_i}{\partial x_i}\\
&= \delta(g\cdot[D, E]).
\end{aligned}$$

Putting $g=c^{-1}$ in (31), we see that $[D, E]$ satisfies (30) in case D and E do, or $c\cdot M_n$ is a Lie algebra. Also

$$(c\cdot M_n)' \subseteq c\cdot M_n'. \tag{32}$$

For suppose that D and E satisfy (30). Writing $g=c^{-1}$, we have

$$\begin{aligned}
&\sum_{j=1}^n \frac{\partial}{\partial x_j}(g\cdot a_i\cdot b_j - g\cdot b_i\cdot a_j)\\
&\qquad = g\cdot\sum_{j=1}^n\left(\frac{\partial a_i}{\partial x_j}\cdot b_j - \frac{\partial b_i}{\partial x_j}\cdot a_j\right) + a_i\cdot\delta(g\cdot E) - b_i\cdot\delta(g\cdot D)\\
&\qquad = g\cdot c_i
\end{aligned}$$

for c_i in (20). Hence $g\cdot[D, E]=(h_1, \cdots, h_n)$ where

$$h_i = \sum_{j \neq i} \frac{\partial}{\partial x_j} (g\cdot a_i \cdot b_j - g\cdot b_i \cdot a_j).$$

Hence $g\cdot [D, E]$ is in the algebra $T_n = S_n$ defined by Frank [3, Lemma 3]. But $S_n = M_n'$ (because S_n is simple in case $n > 2$ and by our earlier remarks in case $n = 2$). Hence $[D, E] \in c\cdot M_n'$, establishing (32). It follows that $c\cdot M_n' = c\cdot S_n$ is a Lie algebra (of dimension $(n-1)(p^n-1)$, the known dimension of S_n), for $(c\cdot M_n')' \subseteq (c\cdot M_n)' \subseteq c\cdot M_n'$ by (32).

Returning to the case $n=2$, we shall prove that $D(A)'' = (\text{ad } A)' \cong V_1$ is a simple Lie algebra of dimension p^2-2 if $c^{-1} \in \tilde{A}$, whereas $D(A)' = \text{ad } A$ is a simple Lie algebra of dimension p^2-1 if $c^{-1} \notin \tilde{A}$. We begin with a normalization of c by proper choice of x_i in A.

THEOREM 5. *Let A be a (simple) nodal noncommutative Jordan algebra of dimension p^2 in K so that multiplication in A is defined by* (28). *Then x_i may be chosen in A so that c is in the form*

$$c = 1 + \alpha x_1^{p-1} \cdot x_2^{p-1}, \qquad \alpha \in F. \tag{33}$$

According as c^{-1} is or is not in $\tilde{A}$ (for any choice of x_i), we have $\alpha = 0$ or $\alpha \neq 0$ in (33).

Proof. Write $c^{-1} = \beta_0 + \beta_1 \cdot x_2 + \cdots + \beta_{p-2}\cdot x_2^{p-2} + \beta_{p-1}\cdot x_2^{p-1}$, $\beta_j \in F[x_1]$. Then β_0^{-1} exists, and $c^{-1}\cdot(1 - \beta_0^{-1}\cdot\beta_{p-1}\cdot x_2^{p-1}) = \beta_0 + \beta_1\cdot x_2 + \cdots + \beta_{p-2}\cdot x_2^{p-2}$. But then (11) and (12) imply that there exists $y_2 = x_2 \cdot h$ such that

$$\frac{\partial y_2}{\partial x_2} = c^{-1}\cdot(1 - \beta_0^{-1}\cdot\beta_{p-1}\cdot x_2^{p-1}). \tag{34}$$

Now $\partial y_2/\partial x_2 = x_2 \cdot \partial h/\partial x_2 + h$ implies that h and β_0 are congruent modulo N. Hence h^{-1} exists, so that $y_2 = \delta x_2 + n$, $n \in N\cdot N$, $\delta \neq 0$. Let $y_1 = x_1$. Then $A = F[y_1, y_2]$, $y_1^p = y_2^p = 0$. Now $y_2^{p-1} = x_2^{p-1}\cdot h^{p-1}$, so that $x_2^{p-1} \in A\cdot y_2^{p-1}$. But every element of $A\cdot y_2^{p-1}$ has the form $\rho\cdot y_2^{p-1}$ for $\rho \in F[y_1]$. Then (3) and (34) imply $[y_1, y_2]/2 = (\partial y_2/\partial x_2)\cdot c = 1 - \beta_0^{-1}\cdot\beta_{p-1}\cdot x_2^{p-1} = 1 + \sigma\cdot y_2^{p-1}$ where

$$\sigma = -\beta_0^{-1}\cdot\beta_{p-1}\cdot\rho \in F[y_1].$$

That is, we may as well take c in (28) in the form

$$c = 1 + \sigma\cdot x_2^{p-1}, \qquad \sigma \in F[x_1]$$

Now $\sigma = \alpha_0 1 + \alpha_1 x_1 + \cdots + \alpha_{p-2} x_1^{p-2} + \alpha_{p-1} x_1^{p-1}$ for $\alpha_j \in F$. Then

$$\begin{aligned} c^{-1}\cdot(1 + \alpha_{p-1} x_1^{p-1}\cdot x_2^{p-1}) &= (1 - \sigma\cdot x_2^{p-1})\cdot(1 + \alpha_{p-1} x_1^{p-1}\cdot x_2^{p-1}) \\ &= 1 - \alpha_0 x_2^{p-1} - \alpha_1 x_1\cdot x_2^{p-1} - \cdots - \alpha_{p-2} x_1^{p-2}\cdot x_2^{p-1}. \end{aligned}$$

By (11) and (12) there exists

$$y_1 = x_1 \cdot (1 + \pi \cdot x_2^{p-1}), \qquad \pi \in F[x_1],$$

such that

$$\frac{\partial y_1}{\partial x_1} = c^{-1} \cdot (1 + \alpha x_1^{p-1} \cdot x_2^{p-1}) \tag{35}$$

where we have written α for $\alpha_{p-1} \in F$. Then $y_1^{p-1} = x_1^{p-1} \cdot (1 - \pi \cdot x_2^{p-1})$, or $x_1^{p-1} = y_1^{p-1} \cdot (1 + \pi \cdot x_2^{p-1})$. Let $y_2 = x_2$. Then $A = F[y_1, y_2]$, while (3) and (35) imply that $[y_1, y_2]/2 = (\partial y_1/\partial x_1) \cdot c = 1 + \alpha x_1^{p-1} \cdot x_2^{p-1} = 1 + \alpha y_1^{p-1} \cdot (1 + \pi \cdot y_2^{p-1}) \cdot y_2^{p-1} = 1 + \alpha y_1^{p-1} \cdot y_2^{p-1}$. That is, we may take c in the form (33).

The final statement in the theorem could probably be established by a careful analysis of the argument above. Instead we note that, for any choice of x_i, V_{01} consists of all $D(g)$ with $g \in A$, and $V_1 = V_{01}'$ of all $D(g)$ for which $g \in \tilde{A}$, while $[D(f), D(g)] = D(h)$ where

$$h = \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \cdot \frac{\partial g}{\partial x_1} \cdot \tag{36}$$

Since 1 is in $\tilde{A}$, as well as being of the form (36), it follows that $\tilde{A}$ consists of all linear combinations of elements of the form (36). By (24) we have $[f, g]/2 = h \cdot c$ for h in (36), or $[A, A] = \tilde{A} \cdot c$. Hence $c^{-1} \in \tilde{A}$ if and only if $1 \in [A, A]$. But the latter condition is independent of the choice of x_i. Hence, for any choice of x_i, $c^{-1} \in \tilde{A}$ if and only if the $c^{-1} = 1 - \alpha x_1^{p-1} \cdot x_2^{p-1}$ given by (33) is in $\tilde{A}$.

One point in the proof of the next theorem is deferred to the final section where we consider the simple Lie algebras $L(G, \delta, f)$.

THEOREM 6. *Let A be a (simple) nodal noncommutative Jordan algebra of dimension p^2 in K so that multiplication in A is defined by* (28). *Then*

$$D(A) = c \cdot M_2 \tag{37}$$

where M_2 is the (p^2+1)-dimensional algebra consisting of all derivations (4) *of B_2 given by* (14) *with $r = 1$, and $D(A)''$ is simple. If $c^{-1} \in \tilde{A}$, then $D(A) \cong M_2$, $D(A)' \cong V_{01}$, and $D(A)'' \cong V_1$ is of dimension $p^2 - 2$. If $c^{-1} \notin \tilde{A}$, then $D(A)' = D(A)'' = c \cdot M_2'$ is of dimension $p^2 - 1$.*

Proof. The case $c^{-1} \in \tilde{A}$ has already been established since we may take $c = 1$ by Theorem 5. Suppose that $c^{-1} \notin \tilde{A}$ so we may take c in the form (33) with $\alpha \neq 0$. We have seen that (37) holds. Hence (32) implies that

$$\dim D(A)' \leqq \dim(c \cdot M_2') = p^2 - 1. \tag{38}$$

We shall see in the next section that a class of central simple Lie algebras L_0 of dimension $p^2 - 1$ is obtained as follows: $L_0 \cong \operatorname{ad} A$ where A has multiplication defined by (28) with $c = \gamma(1 + x_1) \cdot (1 + x_2)$ for $\gamma \neq 0 \in F$. We trace through the steps of the proof of Theorem 5 to see that x_i may be chosen in this A so that

$$\alpha = -\gamma^{p-1} \tag{39}$$

in (33): $c^{-1}=\gamma^{-1}(1+x_1)^{-1}\cdot(1+x_2)^{-1}=\gamma^{-1}(1+x_1)^{-1}(1-x_2+x_2^2- \cdots +x_2^{p-1})$ so that $\beta_0=\beta_{p-1}=\gamma^{-1}(1+x_1)^{-1}$. Then $y_1=x_1$,

$$y_2 = \gamma^{-1}(1+x_1)^{-1}\cdot x_2\cdot\left(1-\frac{1}{2}x_2+\cdots-\frac{1}{p-1}x_2^{p-2}\right),$$

$$y_2^{p-1} = \gamma^{-(p-1)}(1+x_1)^{-(p-1)}\cdot x_2^{p-1}\cdot\left(1-\frac{1}{2}x_2+\cdots-\frac{1}{p-1}x_2^{p-2}\right)^{p-1}$$

$$= \gamma^{-(p-1)}(1+x_1)^{-(p-1)}\cdot x_2^{p-1},$$

so that $x_2^{p-1}=\gamma^{p-1}(1+x_1)^{p-1}\cdot y_2^{p-1}=\gamma^{p-1}(1+y_1)^{p-1}\cdot y_2^{p-1}$. Then α in (33) is the coefficient of $y_1^{p-1}\cdot y_2^{p-1}$ in $-\beta_0^{-1}\cdot\beta_{p-1}\cdot x_2^{p-1}=-x_2^{p-1}$; that is, we have (39). Let $H=F(\gamma)$ where γ satisfies (39). Then $(\text{ad } A)_H=\text{ad}(A_H)\cong L_0$ (an algebra defined over H) is simple and of dimension p^2-1 over H. Hence ad A is simple and of dimension p^2-1 over F. But ad $A\subseteq D(A)$ implies $p^2-1 =\dim(\text{ad } A) = \dim(\text{ad } A)' \leqq \dim D(A)' \leqq p^2-1$ by (38). Hence ad $A =D(A)'=D(A)''=(c\cdot M_2)'=c\cdot M_2'$.

We remark that equality holds in (32) for $n=2$.

4. **The simple algebras** $L(G, \delta, f)$. The simple Lie algebras L_0 and L_δ of Albert and Frank [1] have been generalized by Block(3) in [2] to an extensive class of simple Lie algebras $L(G, \delta, f)$. Block has shown [2, Lemma 3] that each V_r is an algebra $L(G, \delta, f)$. In this section we prove

THEOREM 7. *For any simple Lie algebra* $L(G, \delta, f)$ (*of characteristic* $\neq 2$) *there exists a simple nodal noncommutative Jordan algebra* A *in* K *such that* A^- *is a Lie algebra and* $L(G, \delta, f)\cong(\text{ad } A)'$, *an ideal in* $D(A)$. *Actually* $L_0 \cong \text{ad } A$.

If $L(G, \delta, f)$ is simple, then $G=G_0+G_1+\cdots+G_m$ is an elementary p-group [2, Theorem 2], so that each G_k may be regarded as an n_k-dimensional vector space over the prime field F_p of characteristic p. The order of G is p^n

(3) I am indebted to Dr. Block for furnishing me with a copy of his excellent dissertation [2] before its publication. My Theorem 7 was suggested by his Lemma 3. The following remarks about [2] may be of interest: (i) each of the algebras $V_{m,\mu}$ is isomorphic to V_m, for $y_i=\mu_i^{-1}x_i$, $y_{m+i}=\mu_i^{-1}x_{m+i}$ $(i=1,\cdots,m)$ implies $\mu_i\partial\phi/\partial x_{m+i}=\partial\phi/\partial y_{m+i}$, $-\mu_i\partial\phi/\partial x_i=-\partial\phi/\partial y_i$, and the coefficient of $(x_1\cdots x_{2m})^{p-1}$ is zero if and only if the coefficient of $(y_1\cdots y_{2m})^{p-1}$ is; (ii) if $F=F_p$, the prime field of characteristic p, then any simple $L(G, \delta, f)$ for which $G_0=0$ is isomorphic to V_m, for [2, Theorem 4] implies that in each $G_i\{h(\delta), h(\beta_1), \cdots, h(\beta_k)\}$ and $\{g(\beta_{k+1}, \cdots, g(\beta_r)\}$ are linearly independent sets (of elements in F) over F_p, so that $k=0$ and $r=k+1=1$, requiring that each G_i be 2-dimensional over F_p and that $L(G, \delta, f)\cong V_m$ by [2, Lemma 3] and (i) above.

where $n=n_0+n_1+\cdots+n_m$, and we write $q_{-1}=0$, $q_k=n_0+n_1+\cdots+n_k$ $(k=0, 1, \cdots, m)$. Let $\sigma_1, \cdots, \sigma_{n_0}$ be any basis for G_0 over F_p. Since $\delta=\delta_0+\delta_1+\cdots+\delta_m$ where $\delta_0=0$ and $\delta_k\neq 0$ in G_k for $k=1, \cdots, m$, we may take a basis $\sigma_{q_{k-1}+1}, \cdots, \sigma_{q_k-1}, \delta_k$ for G_k over F_p $(k=1, \cdots, m)$. But then, defining σ_{q_k} by

$$\delta_k = \sigma_{q_{k-1}+1} + \cdots + \sigma_{q_k-1} + \sigma_{q_k}, \tag{40}$$

we also have $\sigma_{q_{k-1}+1}, \cdots, \sigma_{q_k}$ a basis for G_k over F_p $(k=1, \cdots, m)$. Then $\sigma_1, \cdots, \sigma_n$ is a basis for G over F_p, and any $\alpha\in G$ may be written uniquely in the form

$$\alpha = \sum_{i=1}^{n} s_i\sigma_i, \qquad s_i \in F_p. \tag{41}$$

Now $L(G, \delta, f)$ is the derived algebra of a Lie algebra L/Fu_0 of dimension p^n-1 over F, where L has a basis consisting of p^n elements u_α in (1-1) correspondence with the elements α of G. By (41) the u_α are in (1-1) correspondence with the n-tuples $(s_1, \cdots, s_n)$, $s_i\in F_p$, and we shall represent the u_α in this way. The skew-symmetric biadditive function $f(\alpha, \beta)$ on G to F may be taken so that $f(\alpha_k, \beta_l)=0$ for $k\neq l$, $\alpha_k\in G_k$, $\beta_l\in G_l$ $(k, l=0, 1, \cdots, m)$. Writing $f(\sigma_i, \sigma_j)=\alpha_{ij}\in F$, we see that

$$\alpha_{ij} = 0 \text{ unless } q_{k-1}+1 \leqq i, j \leqq q_k \text{ for some } k \quad (0 \leqq k \leqq m). \tag{42}$$

Now (41) and $\beta=\sum_{j=1}^{n} t_j\sigma_j$ imply $f(\alpha_k, \beta_k)=\sum_{i,j=q_{k-1}+1}^{q_k} s_it_j\alpha_{ij}$. Since $\delta=\sum_{k=n_0+1}^{n}\sigma_k$ by (40), we see that [2, (4)] defines multiplication in L by

$$\begin{aligned}(s_1, \cdots, s_n)(t_1, \cdots, t_n) &= \sum_{i,j=1}^{n_0} s_it_j\alpha_{ij}(s_1+t_1, \cdots, s_n+t_n)\\ &+ \sum_{k=1}\Bigg(\sum_{i,j=q_{k-1}+1}^{q_k} s_it_j\alpha_{ij}(s_1+t_1, \cdots, s_{q_{k-1}}+t_{q_{k-1}}, s_{q_{k-1}+1}+t_{q_{k-1}+1}-1,\\ &\qquad \cdots, s_{q_k}+t_{q_k}-1, s_{q_k+1}+t_{q_k+1}, \cdots, s_n+t_n)\Bigg).\end{aligned} \tag{43}$$

Instead of the nilpotent generators x_i of $B_n=F[x_1, \cdots, x_n]$ used in previous sections, we use at this point generators $z_i=1+x_i$ $(i=1, \cdots, n)$. We have $z_i^p=1$, and every element of B_n may be written uniquely in the form

$$f = \sum_{s_i\in F_p} \alpha_{s_1\cdots s_n} z_1^{s_1}\cdot \cdots \cdot z_n^{s_n}, \qquad \alpha_{s_1\cdots s_n} \in F. \tag{44}$$

Let A in K be of dimension p^n so that $A^+=B_n$. Then (24) implies that multiplication in A^- is defined by

$$[f, g] = \sum_{i,j} \frac{\partial f}{\partial z_i}\cdot\frac{\partial g}{\partial z_j}\cdot 2c_{ij}$$

since $\partial f/\partial z_i = \partial f/\partial x_i$. Equivalently, multiplication in A^- is defined by

$$[z_1^{s_1}\cdot \cdots \cdot z_n^{s_n},\ z_1^{t_1}\cdot \cdots \cdot z_n^{t_n}] \tag{45}$$
$$= \sum_{i,j=1}^{n} s_i t_j z_1^{s_1+t_1}\cdot \cdots \cdot z_i^{s_i+t_i-1}\cdot \cdots \cdot z_j^{s_j+t_j-1}\cdot \cdots \cdot z_n^{s_n+t_n}\cdot 2c_{ij}.$$

Let

$$\begin{aligned} c_{ij} &= 0 \qquad \text{unless } q_{k-1}+1 \leqq i, j \leqq q_k \text{ for some } k\ (0 \leqq k \leqq m),\\ 2c_{ij} &= \alpha_{ij} z_i \cdot z_j \qquad \text{for } 1 \leqq i, j \leqq n_0,\\ 2c_{ij} &= \alpha_{ij} z_i \cdot z_j \cdot (z_{q_{k-1}+1}\cdot \cdots \cdot z_{q_k})^{-1}\\ &\qquad \text{for } q_{k-1}+1 \leqq i, j \leqq q_k\ (k = 1, \cdots, m). \end{aligned} \tag{46}$$

For typographical reasons we write $\{s_1, \cdots, s_n\}$ for $z_1^{s_1}\cdot \cdots \cdot z_n^{s_n}$. Then (45) and (46) imply

$$\begin{aligned} [\{s_1, \cdots, s_n\}, \{t_1, \cdots, t_n\}] &= \sum_{i,j=1}^{n_0} s_i t_j \alpha_{ij}\{s_1+t_1, \cdots, s_n+t_n\}\\ &+ \sum_{k=1}^{n}\Bigg(\sum_{i,j=q_{k-1}+1}^{q_k} s_i t_j \alpha_{ij}\{s_1+t_1, \cdots, s_{q_{k-1}}+t_{q_{k-1}}, s_{q_{k-1}+1}+t_{q_{k-1}+1}-1, \cdots,\\ &\qquad s_{q_k}+t_{q_k}-1, s_{q_k+1}+t_{q_k+1}, \cdots, s_n+t_n\}\Bigg). \end{aligned} \tag{47}$$

That is, $L \cong A^-$ by (43) and (47), and $L/Fu_0 \cong A^-/F1$.

In order to complete the proof of the theorem, we shall require the following

Lemma 2. *Let A be a noncommutative Jordan algebra such that A^+ is associative. If $[f, g]=0$ for every $g \in A$, then $f\cdot A$ $(=fA=Af)$ is an ideal of A. Thus, if A is any simple nodal noncommutative Jordan algebra, $[f, g]=0$ for every $g\in A$ if and only if $f \in F1$.*

Proof. Clearly $fg=gf=f\cdot g$ for every $g\in A$. Then [6, (4)] implies $(f\cdot g)h = -(g\cdot h)f + (gh)\cdot f + (gf)\cdot h = -f\cdot g\cdot h + f\cdot(gh) + f\cdot g\cdot h = f\cdot (gh) \in f\cdot A$. If $A = F1+N$ is simple, then either $f\cdot A=0$, implying $f=0$, or $f\cdot A = A$. In the latter case $f=\alpha 1+z$, $\alpha\neq 0$, $z\in N$. But then $[z, g]=[f-\alpha 1, g]=0$ for every g, implying $z=0$, $f=\alpha 1\in F1$.

We return to the proof of Theorem 7. A is in K, since at least one of the α_{ij} in (46) is not zero [2, Theorem 2]. If $G=G_0$, then $L_0 = L(G, \delta, f) = L/Fu_0$ has dimension p^n-1. If $G\neq G_0$, then $L(G, \delta, f) = (L/Fu_0)'$ has dimension p^n-2. We shall show in both cases that A is simple since $L(G, \delta, f)$ is. If A is not simple, then A has a nonzero ideal $B\subseteq N$. Since N is not an ideal of A,

$$1 \leqq \dim B \leqq p^n - 2. \tag{48}$$

B^- is an ideal of A^-, and either $F1\cap B^- = F1$, implying $1\in B$, $B=A$, a contradiction, or $F1\cap B^-=0$. Hence $C^-=F1\oplus B^-$ is an ideal of A^-. In case $G=G_0$, then $A^-/F1\cong L(G, \delta, f)$ is simple, so the kernel $C^-/F1$ of the natural homomorphism of $A^-/F1$ onto A^-/C^- is either 0 or all of $A^-/F1$. That is, either $B^-=0$ or dim $B^-=p^n-1$, contradicting (48) in either event. In case $G\neq G_0$, $L/Fu_0=L(G, \delta, f)+Fv$ contains an ideal M corresponding to $C^-/F1$ in $A^-/F1$. Clearly $M\cong B^-$. Then $M\cap L(G, \delta, f)$ is an ideal of the simple algebra $L(G, \delta, f)$. In view of (48), it follows that either (i) $M=L(G, \delta, f)$, or (ii) $L/Fu_0=L(G, \delta, f)\oplus M$ where dim $B=$dim $M=1$. In case (i), $L/Fu_0=M+Fv$. Correspondingly, $A^-/F1=C^-/F1+F\bar{z}$ where z may be taken to be in N. Then $A=C+Fz=F1+B+Fz$, $N=B+Fz$. Now B an ideal of A implies $NN=(B+Fz)(B+Fz)\subseteq B+Fz^2\subseteq N$, a contradiction, since A is a nodal algebra. There remains the possibility (ii), $L/Fu_0=(L/Fu_0)'\oplus M$. Correspondingly, $(A^-/F1)=(A^-/F1)'\oplus F\bar{w}$ where w may be taken in B. Then $B=Fw$. Since B^+ is a 1-dimensional ideal in $A^+=F[x_1, \cdots, x_n]$, we have $w=\sigma x_1^{p-1}\cdot \cdots \cdot x_1^{p-1}$ for $\sigma\neq 0\in F$. Write $c_{ij}=\alpha_{ij}1+z_{ij}$ in (24), $\alpha_{ij}\in F$, $z_{ij}\in N$. There exist i_0, j_0 such that $\alpha_{i_0j_0}\neq 0$. Then $[x_{i_0}, x_1^{p-1}\cdot \cdots \cdot x_n^{p-1}]\in B$ implies

$$0\equiv [x_{i_0}, x_1^{p-1}\cdot \cdots \cdot x_n^{p-1}]\equiv (p-1)\sum_j x_1^{p-1}\cdot \cdots \cdot x_j^{p-2}\cdot \cdots \cdot x_n^{p-1}\cdot 2c_{i_0j}$$
$$\equiv 2(p-1)\sum_j \alpha_{i_0j}x_1^{p-1}\cdot \cdots \cdot x_j^{p-2}\cdot \cdots \cdot x_n^{p-1} \bmod B = Fx_1^{p-1}\cdot \cdots \cdot x_n^{p-1},$$

implying $\alpha_{i_0j_0}=0$, a contradiction. That is, A must be simple if $L(G, \delta, f)$ is. That ad A is isomorphic to $A^-/F1$ follows directly from Lemma 2.

In §2 we referred to the proof above for justification of the statement that any A defined by a nondegenerate form ϕ is simple. This follows from the fact that $V_r=L(G, \delta, f)$ where $G_0=0$, $m=r$, and G_k is of dimension 2 over F_p for $k=1, \cdots, r$ [2, Lemma 3]. For then the c_{ij} defined by (46) are all in $F1$. In the proof of Theorem 6 we relied on (46) for the case $n=n_0=2$. In that instance $2c=2c_{12}=\alpha_{12}z_1\cdot z_2=\alpha_{12}(1+x_1)\cdot(1+x_2)$ with $\alpha_{12}\neq 0$.

We have not computed the derivations of the algebras A in Theorem 7. Instead we conclude with the following result which generalizes (26) in the direction of (14).

THEOREM 8. *Let A be in K so that multiplication is defined by* (1). *If A^- is a Lie algebra, then the mappings D defined by* (4) *with*

$$a_i=\sum_{j=1}^{n}\left(\frac{\partial g}{\partial x_j}+\alpha_j x_j^{p-1}\right)\cdot c_{ij}, \qquad i=1, \cdots, n, \tag{49}$$

for any $g\in A$ and any $\alpha_j\in F$ $(j=1, \cdots, n)$, are derivations of A.

Since ad $g/2$ in (25) is a derivation of A, it is sufficient to verify that D in (4) is a derivation in case

$$a_i=\sum_{k=1}^{n}\alpha_k x_k^{p-1}\cdot c_{ik}, \qquad \alpha_k\in F. \tag{50}$$

Now D is a derivation of A in case (5) with k replaced by t is satisfied. But (50) implies

$$\sum_{t=1}^{n}\left(\frac{\partial c_{ij}}{\partial x_t}\cdot a_t + \frac{\partial a_i}{\partial x_t}\cdot c_{jt} + \frac{\partial a_j}{\partial x_t}\cdot c_{ti}\right)$$

$$= \sum_{k,t}\left(\frac{\partial c_{ij}}{\partial x_t}\cdot \alpha_k x_k^{p-1}\cdot c_{tk} + \alpha_k \frac{\partial(x_k^{p-1}\cdot c_{ik})}{\partial x_t}\cdot c_{jt} + \alpha_k \frac{\partial(x_k^{p-1}\cdot c_{jk})}{\partial x_t}\cdot c_{ti}\right)$$

$$= \sum_{k} \alpha_k x_k^{p-1}\cdot \left\{ \sum_{t}\left(\frac{\partial c_{ij}}{\partial x_t}\cdot c_{tk} + \frac{\partial c_{ik}}{\partial x_t}\cdot c_{jt} + \frac{\partial c_{jk}}{\partial x_t}\cdot c_{ti}\right)\right\}$$

$$+ \sum_{k} \alpha_k x_k^{p-2}\cdot (c_{ik}\cdot c_{jk} + c_{jk}\cdot c_{ki})$$

$$= 0$$

by (27).

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